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STABILIZATION OF THE MOTIONS OF MECHANICAL SYSTEMS WITH NON-HOLONOMIC CONSTRAINTS†

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Mechanical systems possibly containing non-holonomic constraints are considered. The problem of stabilizing the motion of the system along a given manifold of its phase space is solved. A control law which does not involve the dynamical parameters of the system is constructed. The law is universal, that is, it stabilizes motion along any given manifold. It is only necessary that the manifold be feasible, that is, conform to the dynamics of the system. © 2000 Elsevier Science Ltd. All rights reserved.

An equilibrium position of a non-holonomic system need not necessarily be isolated, and one is usually dealing with a manifold of equilibrium positions [1–4]. Stability conditions appropriate to this situation have been investigated, and various control problems relating to the stabilization of manifolds of equilibrium positions in non-holonomic systems have been considered [8, 9].

In this paper, we consider the stabilization of manifolds of general form, not necessarily corresponding to an equilibrium position of the system. A control law is constructed on the basis of Appell's equations for the motion of a non-holonomic system. It is obtained in the class of discontinuous feedbacks. The law does not explicitly involve the dynamical parameters of the mechanical system (the mass, moments of inertia, friction coefficients, etc.). This implies that the control law is universal [10–15], in the sense that it stabilizes not one but any given manifold of the system. The only necessary condition is that the manifold be feasible, that is, conform essentially to the dynamics of the controlled object (in particular, to the limited dynamical possibilities of its control devices).

1. FORMULATION OF THE PROBLEM

Consider a controlled mechanical system of general form with m generalized coordinates, m controls and g non-holonomic constraints [1–4]. As an example of such a controlled system one can think of a mechanical system with rolling (a multisection controlled suspension of a pneumatic wheel) [4] or a system included in a machining process. It may be the complex multisection controlled fitting of a cutting tool, or a manipulator with a special machining device. It is assumed that the machining process may involve non-holonomic constraints.

To describe the motion of a non-holonomic system of this type, one uses the following system of equations, due to Appell

$$\partial U/\partial\ddot{\pi}_s = \Pi_s \tag{1.1}$$

$$\sum_{i=1}^{m} f_{si}(q,\ t)\dot{q}_i + f_s(q,\ t) - \dot{\pi}_s = 0 \tag{1.2}$$

$$\sum_{i=1}^{m} f_{pi}(q,\ t)\dot{q}_i + f_p(q,\ t) = 0 \tag{1.3}$$

This system consists of $n + m$ differential equations in m generalized coordinates q_i and $n(n = m - g)$ quasi-velocities of the system $\dot{\pi}_s$ [1–4]. Throughout this paper it is assumed that the indices s, r, q run through values $1, 2, \ldots, n$; $i, j, k - 1, 2, \ldots, m, p - n + 1, n + 2, \ldots, m$.

Subsystem (1.1) comprises Appell's equations proper; subsystem (1.2) can be regarded as relations defining the quasi-velocities $\dot{\pi}_s$ of a system with generalized variables q_i, $\dot{q}_i$, and (1.3) can be regarded as a description of non-holonomic mechanical constraints (this means that the system of differential constraints (1.3) is not reducible to any system of geometrical constraints $f_p(q, t) = 0$ [1–4]).

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The notation used in system (1.1)–(1.3) is the generally accepted standard one [1–4]. In addition, $U = U(q, \dot{\pi}, \ddot{\pi}, t)$ denotes the acceleration function of the mechanical system and $\Pi_s(q, \dot{\pi}, t)$ are the generalized forces corresponding to quasi-coordinates π_s

$$\Pi_s = \sum_i (Q_i + M_i) b_{is} \tag{1.4}$$

The quantities $(Q_i + M_i)$ in (1.4) are the generalized forces of the mechanical system corresponding to the generalized coordinates q_i and M_i denotes the generalized control forces generated by the control devices of the system; these forces are assumed to be bounded

$$|M_i| \leqslant H_i, \quad H_i = \text{const} > 0 \tag{1.5}$$

Thus, the mechanical system under consideration has a control for each coordinate q_i, as is typical, for example, in manipulator robots [10–15].

The quantities $b_{is}(q, t)$ in (1.4) are the coefficients of the matrix $B = \| b_{ik} \|$, which satisfies the relations

$$FB = E, \quad \text{rank}\, F = m \tag{1.6}$$

whence

$$\dot{q}_i = \sum_s b_{is} \dot{\pi}_s + b_i \tag{1.7}$$

where $F = \|f_{ki}\|$, f_{ki} as in (1.2) and (1.3), and E is the identity matrix. Condition (1.6) is generally assumed in the dynamics of non-holonomic mechanical systems [1–9].

Equations (1.1) may be written in the following form [1–4]

$$\sum_r u_{sr} \ddot{\pi}_r = \sum_i (R_i + M_i) b_{is} \tag{1.8}$$

where u_{sr} are the coefficients of the matrix $u = \|u_{sr}(q, t)\|$

$$u_{sr} = \partial^2 U / \partial \ddot{\pi}_s \partial \ddot{\pi}_r \tag{1.9}$$

and U is the acceleration function; the function $R_i = R_i(q, \dot{\pi}, t)$ is constructed from the expression for $\partial U / \partial \ddot{\pi}_s$ in (1.1) and the generalized forces Q_i in (1.4) [1–4].

To analyse system (1.8), we introduce the following condition

$$|u_{ik}(q, t)| \leqslant D, \quad |R_i(q, \dot{\pi}, t) \leqslant D, \quad \ldots \tag{1.10}$$

It is assumed that this condition holds for all functions $u_{ik}(q, t)$, $R_i(q, \dot{\pi}, t)$ and $f_{ik}(q, t)$, $f_i(q, t)$, $b_{ik}(q, t)$, $b_i(q, t)$ and their derivatives, where the number $D > 0$ may be fairly large. Note that condition (1.10) requires the functions to be bounded only in principle and is not restrictive. This condition seems natural in the dynamics of mechanical systems [1–12]. It may be substantially weakened, requiring only that the functions be smooth [10–15].

The objective of the control of system (1.1)–(1.3) is given in the form

$$\dot{\pi}_s = 0 \tag{1.11}$$

or, taking (1.2) and (1.3) into consideration, in the form

$$\sum_i f_{ki}(q, t) \dot{q}_i + f_k(q, t) = 0 \tag{1.12}$$

In other words, the objective of the control is to make the motion of the mechanical system take place along the given manifold (1.12) of its phase space $(q_i, \ldots, q_m, \dot{q}_1, \ldots \dot{q}_m)$ [10–15]. The idea is, through the choice of the functions $f_{si}(q, t)$, $f_s(q, t)$ in (1.12), to ensure satisfaction of meaningful control objectives in a mechanical system.

In fact, Eqs (1.12) enable one to describe the control objectives in mechanical systems in the most general form [10]. Suppose, for example, that the control objective in a manipulator is to make its gripping device follow a prescribed path in space, with a given orientation. This objective may be described by the following relations [10]

$$g_l(q, t) = 0, \quad l = 1, 2, \ldots, 6 \tag{1.13}$$

There are methods for constructing system (1.2) for which (1.13) is a stable manifold [16]. A control objective corresponding to steering the system to some final configuration may also be given the form (1.12). When that is done allowance may also be made for additional requirements, such as maintaining a certain velocity regime in the elements of the mechanical system. Restrictions on the motion of the system (phase restrictions), which arise when allowance is made for the positions and velocities of surrounding objects (obstacles), may also be represented in an analogous form [10–15].

The control problem for system (1.1)–(1.3) is to construct admissible controls M_i (that is, controls satisfying (1.5)) under which Eqs (1.12) describe a stable manifold of the system in accordance with the following definition.

Definition 1. Equations (1.12) describe a stable manifold of system (1.1)–(1.3) if, for any $\varepsilon > 0$, a $\delta = \delta(\varepsilon) > 0$ exists such that the inequalities

$$\left|\sum_i f_{ki}(q, t)\dot{q}_i + f_k(q, t)\right| \leqslant \varepsilon, \quad t \geqslant t^0 \tag{1.14}$$

hold for $t \geqslant t^0$ in any motion of system (1.1)–(1.3), provided that, at time $t = t^0$

$$\left|\sum_i f_{ki}(q^0, t^0)\dot{q}_i^0 + f_k(q^0, t^0)\right| \leqslant \delta, \quad \forall q^0, \dot{q}_i^0 \tag{1.15}$$

The problem is to construct a control law for system (1.1)–(1.3), in the form

$$M_i = M_i(f_{lj}(q, t), \dot{q}_k, f_l(q, t)) \tag{1.16}$$

which does not involve the functions $u_{sr}(q, t)$, and $Q_i(q, \dot{q}, t)$. The law must ensure that the motion of the system along a given manifold (1.12) is stable. It must be admissible, that is, it must satisfy conditions (1.5). It must also be universal [10–15], that is, under any substitution $f_{sj} \to \tilde{f}_{sj}$, $f_s \to \tilde{f}_s$, the law (1.16) must ensure stabilization of the new manifold $\Sigma_j \tilde{f}_{kj}\dot{q}_j + \tilde{f}_k = 0$ instead of (1.12), provided only that the motion of the system along this manifold is feasible in practice (i.e. essentially, physically feasible) [11–15].

2. FEASIBLE MANIFOLDS OF A MECHANICAL SYSTEM

The control objective must conform to the dynamical possibilities of the controlled object (1.1)–(1.3). Namely, the motion of system (1.12) must also be a motion of the mechanical system under consideration, that is, system (1.1)–(1.3), (1.12) must be consistent.

To formalize this condition, we will consider Eqs (1.12) as a system of m differential equations, describing the variation of the m generalized coordinates of the system q_i($q_i(t)$ is a solution of system (1.12)). We introduce the set Φ of all possible motions of this system [11–15]. To fix our ideas, we will include in Φ all possible solutions $q = q(t)$ of system (1.12) that begin at time $t = t^0$ with initial data $q(t^0) = q^0$ from some admissible domain

$$|q_i^0| \leqslant d, \quad d = \text{const} > 0$$

Consider a motion $(q, \dot{\pi}) = (q, 0)$ of system (1.1)–(1.3), which corresponds to a motion q of system (1.12). The motions $q(t)$ of system (1.12) are smooth (this follows from (1.10)). Henceforth, therefore, we will consider smooth motions $(q, 0)$ of system (1.1)–(1.3) on the assumption that $\ddot{\pi} = 0$, when it takes the form

$$\sum_i (R_i(q, 0, t) + M_i^0(t))b_{is}(q, t) = 0, \quad M_i^0(t) = M_i\Big|_{\dot{\pi}=0} \tag{2.1}$$

$$\sum_i f_{ki}(q, t)\dot{q}_i + f_k(q, t) = 0 \tag{2.2}$$

Equations (2.2) are identical with (1.12); Eqs (2.1) may be regarded as defining the generalized forces $M_i = M^0$ of the mechanical system in unperturbed motion $(q, 0)$ along the manifold (1.12).

Admissible controls of the mechanical system must be bounded, by (1.5). This means that in unperturbed motion of system (1.1)–(1.3) along system (1.12) (conforming to the control objective), Eqs (2.1) and (1.5) must hold, in the following form

$$\sum_i R_i^0 b_{is} = -\sum_i M_i^0 b_{is}, \quad |M_i^0| \leq H_i \tag{2.3}$$

or

$$\left|\sum_i R_i^0 b_{is}\right| \leq \sum_i H_i |b_{is}| \tag{2.4}$$

where the functions $R^0{}_i$, b_{is} depend on q_i and t

$$R_i^0 = R_i(q, 0, t) \tag{2.5}$$

Inequalities (2.4) we have constructed must be compatible with equations (1.12). In other words, the phase constraints (2.4) must be satisfied in all motions of system (1.12)

$$\left[\left|\sum_i R_i^0(q, t) b_{is}(q, t)\right| - \sum_i H_i |b_{is}(q, t)|\right]\Bigg|_{q\in\Phi} \leq 0 \tag{2.6}$$

When condition (2.6) is satisfied, system (1.1)–(1.3), (1.5), (1.12) is consistent. This means that a control in domain (1.5) exists, which forces the motion of the mechanical system (1.1)–(1.3) to take place along the given manifold (1.12). In what follows we shall say that such a manifold is feasible. If condition (2.6) is not satisfied, the system specified is not consistent, that is, motion of system (1.1)–(1.3), (1.5) along the manifold (1.12) need not be possible (the manifold (1.12) is not feasible). Henceforth, therefore, we will assume that conditions (2.6) are satisfied.

3. STABILIZATION OF MANIFOLDS

To stabilize a manifold (1.12), we introduce discontinuous controls [10–15, 17]

$$M_i = M_i^1 = -H_i \operatorname{sign}\left(\sum_s b_{is}\dot{\pi}_s\right) \tag{3.1}$$

with b_{is} as in (1.6). Taking note of (3.1) and (1.7), we rewrite subsystem (1.1) ((1.8)) in the form

$$\sum_r u_{sr}\ddot{\pi}_r = \sum_i (R_i - H_i \operatorname{sign}(\dot{q}_i - b_i)) b_{is} \tag{3.2}$$

Theorem 1. Any manifold (1.12) is an exponentially stable manifold of system (3.2), (1.2), (1.3) (by analogy with Definition 1) if it is feasible in the sense that the following form of conditions (2.6) holds

$$|R_i^0(q(t), t)|_{q\in\Phi} \leq H_i - \eta \tag{3.3}$$

where the constant $\eta > 0$ may be fairly small, and conditions (1.6) and (1.10) hold.

The proof of Theorem 1 will be presented below in Section 4.

The statement of Theorem 1 generalizes well-known results [10–15] in two main areas: (1) mechanical systems (1.1)–(1.3) of general form (containing non-holonomic constraints) are considered; (2) the control objective for the system is defined in the general form (1.12).

Theorem 1 is essentially stating that control law (3.1) as constructed will ensure the stability of practically any feasible manifold of the non-holonomic system. The law is universal [10–15], since under the substitution $f_{sj} \to \tilde{f}_{sj}$, $f_s \to \tilde{f}_s$ (that is, $b_{is} \to \tilde{b}_{is}$, $b_i \to \tilde{b}_i$ in (3.1)) it stabilizes the new given manifold $\Sigma_j \tilde{f}_{sj}\dot{q}_j + \tilde{f}_s = 0$ instead of (1.12). The only precondition is that this manifold be feasible, that is, condition (2.6) hold in the form (3.3). This is possible owing to the fact that the control law does not explicitly involve the functions $u_{sr}(q, t)$, $Q_i(q, \dot{q}, t)$, that is, it does not involve the dynamical parameters of the controlled object (mass, moments of inertia, friction coefficients, etc.). The dependence on these parameters is implicit, through the feasibility condition (2.6).

Note that condition (3.3) of Theorem 1 is not substantially stronger than feasibility condition (2.6). In fact, if the matrix $B = \| b_{ik} \|$ is diagonal and $\eta = 0$, then these conditions are identical. The number η in (3.4) may be chosen to be fairly small.

We also note that inequalities (3.3), and hence also (2.6), are satisfied for sufficiently large H_i in (1.5). Indeed, let us introduce the notation

$$h_i = \max_{q\in\Phi} | R_i^0(q(t),\ t) |, \quad \bar{h}_i = \max_{q,\ t} | R_i^0(q,\ t) | \tag{3.4}$$

where $\bar{h}_i < \infty$ in (1.10). Inequalities (3.3) follow from the conditions

$$h_i \leqslant \bar{h}_i \leqslant H_i \tag{3.5}$$

4. A RIGOROUS PROOF OF THE STABILITY OF MANIFOLDS

Theorem 1 is proved using the scheme of [10–15], which developed the direct method of Lyapunov functions as applied to mechanical systems. Namely, we introduce a function which will henceforth play the role of a Lyapunov function

$$G = \frac{1}{2} \sum_{r,s} u_{rs} \dot{\pi}_r \dot{\pi}_s \tag{4.1}$$

Let u_{rs} denote the coefficients of the matrix $u = \| u_{rs}(q, t) \|$ introduced in (1.9). This function G satisfies the inequalities

$$\lambda_1 \sum_s \dot{\pi}_s^2 \leqslant G(q,\ \dot{\pi},\ t) \leqslant \lambda_2 \sum_s \dot{\pi}_s^2 \tag{4.2}$$

where λ_i are numbers such that $0 < \lambda_1 \leqslant \lambda_2 < \infty$; these inequalities follow from (1.10) [1–4, 10–15].

In accordance with the basic positions of [10–15], the purpose of the controls is to make the function G decrease to zero, e.g. in accordance with the inequality

$$\dot{G}(q,\ \dot{\pi},\ t,\ M) < 0 \tag{4.3}$$

which is Lyapunov's condition for stability of motion ($\dot{G}(q, \dot{\pi}, t, M)$ is the derivative of G along the trajectories of the control object (3.2)).

The derivative

$$\dot{G} = \sum_{r,s} \dot{\pi}_s u_{rs} \ddot{\pi}_r + \frac{1}{2} \sum_{r,s} \dot{u}_{rs} \dot{\pi}_r \dot{\pi}_s \tag{4.4}$$

along trajectories of (3.2) has the form

$$\dot{G} = \sum_s \dot{\pi}_s \left(\sum_i (R_i + M_i^1) b_{is} + \frac{1}{2} \sum_r \dot{u}_{rs} \dot{\pi}_r \right) \tag{4.5}$$

Inequality (4.3) is obtained from expression (4.5) by majorizing the right-hand side, taking into account the properties of the mechanical system under consideration.

To that end, we write (4.5) in fully developed form

$$\dot{G} = \sum_s \dot{\pi}_s \left(\sum_i (R_i + M_i^1) b_{is} + \frac{1}{2} \sum_r \left[\sum_j \frac{\partial u_{rs}}{\partial q_j} \dot{q}_j + \frac{\partial u_{rs}}{\partial t} \right] \dot{\pi}_r \right) \tag{4.6}$$

By (1.7), we can write (4.6) in the form

$$\dot{G} = \sum_s \dot{\pi}_s \left(\sum_i (R_i + M_i^1) b_{is} + \sum_{q,r} \dot{\pi}_r \left[A_{sr} + B_{srq} \dot{\pi}_q \right] \right) \tag{4.7}$$

where A_{sr} and B_{srq} are certain functions of q_j and t.

Since G satisfies inequalities (4.2), we obtain the following inequality from (4.7)

$$\dot{G} \leqslant \sum_s \dot{\pi}_s \sum_i (R_i + M_i^1) b_{is} + G(a + b\sqrt{G}) \tag{4.8}$$

where inequalities (1.10) have been taken into consideration and a and b are non-negative numbers.

We will now show that the right-hand side of inequality (4.8) is negative-definite, that is, the desired inequality (4.3) will follow from (4.8). To do this, we use (2.5) and write inequality (4.8) in the form

$$\dot{G} \leqslant \sum_i \left[R_i^0 + M_i^1\right]\sigma_i + G(\bar{a} + \bar{b}\sqrt{G}), \quad \sigma_i = \sum_s b_{is}\dot{\pi}_s \tag{4.9}$$

Taking (3.1) into account, we can write this inequality in the form

$$\dot{G} \leqslant \sum_i \left[R_i^0 - H_i \operatorname{sign}(\sigma_i)\right]\sigma_i + G(\bar{a} + \bar{b}\sqrt{G}) \tag{4.10}$$

or

$$\dot{G} \leqslant -\sum_i \left[-R_i^0 \operatorname{sign}(\sigma_i) + H_i\right] |\sigma_i| + G(\bar{a} + \bar{b}\sqrt{G}) \tag{4.11}$$

Note that the choice of the control in the form (3.1) satisfies the general principle for constructing controls [10]. Indeed, the control M_i^1 of (3.1) gives the derivative $\dot{G}$ its minimum value in the domain of admissible controls (1.5) [5–7]. In other words, expression (4.5) for $\dot{G}$ may be written in the form

$$\dot{G}(q, t, \dot{\pi}, M^1) = \min_{|M_i| \leqslant H_i} \dot{G}(q, t, \dot{\pi}, M) \tag{4.12}$$

Taking condition (3.3) of Theorem 1 into consideration, we can rewrite inequality (4.11) in the form

$$\dot{G} \leqslant -\eta \sum_i |\sigma_i| + G(\bar{a} + \bar{b}\sqrt{G}) \tag{4.13}$$

The sum in (4.13) satisfies the inequalities

$$\sum_i |\sigma_i| = \sum_i \left|\sum_s b_{is}\dot{\pi}_s\right| \geqslant \gamma_1 \sum_s |\dot{\pi}_s| \geqslant \gamma_2\sqrt{G} \tag{4.14}$$

where $\gamma_1 = \text{const} > 0$, $i = 1, 2, \ldots$. Indeed, using (1.7), we can rewrite (1.2) in the form

$$\dot{\pi}_s = \sum_i f_{si}[\sigma_i - b_i] + f_s = \sum_i f_{si}[\sigma_i] \tag{4.15}$$

Hence it follows that

$$\sum_s |\dot{\pi}_s| \leqslant \sum_s \sum_i |f_{si}||\sigma_i| \tag{4.16}$$

In view of condition (1.10) in Theorem 1, inequality (4.16) implies the first inequality in (4.14). The second follows directly from (4.2). Arguing from inequalities (4.13) and (4.14), we obtain inequality (4.3) in the form

$$\dot{G} \leqslant -\bar{\eta}\sqrt{G} + G(\bar{a} + \bar{b}\sqrt{G}) \tag{4.17}$$

We will now show explicitly that the solutions $G = G(t)$ of inequality (4.17) tend exponentially to zero. This will imply, in view of (4.2), the analogous property for a motion $\dot{\pi}_s = 0$ of system (3.2) under consideration, thus proving the main part of Theorem 1.

The motion $G = 0$ of system (4.17) is stable.

Indeed, in the domain

$$-\bar{\eta} + \sqrt{G}(\bar{a} + \bar{b}\sqrt{G}) \leqslant -\tfrac{1}{2}\bar{\eta} \tag{4.18}$$

inequality (4.17) follows from the inequality

$$\dot{G} \leqslant -\tfrac{1}{2}\sqrt{G}\bar{\eta} \tag{4.19}$$

The solutions of this inequality are [10–15]

$$G(t) \leqslant G(t^0), \quad \text{for} \quad t^0 \leqslant t \leqslant t^1, \quad G(t) = 0, \quad \text{for} \quad t \geqslant t^1 \tag{4.20}$$

$$t^1 = \frac{1}{\eta} 4\sqrt{G(t^0)} \tag{4.21}$$

Relations (4.20) express the fact that the solution $G = 0$ of system (4.17) is stable.

Now suppose that the motion of system (4.17) begins in domain (4.18). Then its motion will remain in that domain. In (4.18), the truth of (4.17) follows from inequality (4.19). This implies the truth of the stability relations (4.20) for system (4.17) also. In view of inequality (4.2), this implies that the solution $\dot{\pi}_s = 0$ of system (4.3) is (exponentially) stable, which is the assertion of Theorem 1. The theorem is proved.

5. STABILIZATION OF THE MOTIONS OF A NON-HOLONOMIC SYSTEM

It follows from Theorem 1 that the quantities $\dot{\pi}_s$ satisfy the following inequalities along motions of system (3.2)

$$|\dot{\pi}_s| \leqslant \gamma |\dot{\pi}_s(t^0)| \exp(-\lambda(t - t^0)) \tag{5.1}$$

where γ and λ are non-negative constants and, by (4.20), for $t \geqslant t^1 \geqslant t^0$, $\dot{\pi}_s = 0$, and system (1.7) takes the form

$$\dot{q}_i = b_i(q, t) \tag{5.2}$$

We stipulate that system (5.2) has some exponentially stable motion $q = q^*(t)$. Then the following theorem holds.

Theorem 2. Under the assumptions of Theorem 1, suppose the motion $q = q^*(t)$ of system (5.2) is exponentially stable. Then $(q^*, 0)$ is an exponentially stable motion of system (3.2).

The proof of Theorem 2 uses a Lyapunov function $g(q, t)$, which exists due to the assumption that the motion $q = q^*(t)$ of system (5.2) is exponentially stable [18]. The derivative of g along trajectories of system (5.1) is

$$\dot{g} = \dot{g}_1 + \sum_i g_i'(q, t)\sigma_i \tag{5.3}$$

where $\dot{g}_1$ is the derivative of g along trajectories of unperturbed system (5.2). The right-hand side of (5.3) can be majorized, taking into account the fact that g satisfies estimates characteristic for quadratic forms [18]. On the basis of these estimates, also using the fact that the perturbing term in (5.3) is a decreasing function, one establishes that g decreases along trajectories of system (1.7), that is its motion q^* is stable, and hence so is the motion q^* of system (3.2).

It follows from Theorem 2 that system (3.2) will move along the given manifold (1.12) with its coordinates q_i close to the give quantities $q^*_i(t)$. The motion $q^*(t)$ may correspond, for example, to an equilibrium position of the mechanical system [5]. Thus, the stability of the equilibrium $q^*(t)$ follows from the assumption that the equilibrium $q^*(t)$ in system (5.2) is stable.

A non-holonomic system need not have a unique equilibrium position, that is, there may be a manifold $S(q) = 0$ of equilibrium positions [4]. We stipulate that $S(q) = 0$ should be a stable manifold of system (5.2). In that case (with suitable additional assumptions) $S(q) = 0$ will be a stable manifold of system (3.2) also.

Thus, the problem of stabilizing the motions of non-holonomic systems [5–9] may essentially be reduced to the question of whether one can construct a system (5.2) with a given stable motion q^* (or manifold $S(q) = 0$) by a suitable choice of the functions $f_{si}(q, r), f_s(q, t)$ in the initial relations (1.2). This question relates to the problem of constructing differential equations with given asymptotically stable integral manifolds [16].

6. STABILITY TO PERSISTENT PERTURBATIONS

The motion of a non-holonomic system will also be stable when allowance is made for various non-ideal features of the measuring and actuating elements of the control system [14, 15]. If the measurement errors ξ_i in the generalized velocities $\dot{q}_i$ are taken into account system (3.2) becomes

$$\sum_r u_{sr}\ddot{\pi}_r = \sum_i (R_i - H_i \operatorname{sign}(\dot{q}_i + \xi_i - b_i))b_{is} \tag{6.1}$$

Suppose the errors are small

$$|\xi_i| \leq \bar{\xi} \tag{6.2}$$

where $\bar{\xi}$ = const > 0 is a small quantity. Then it may be shown that Theorem 1 remains valid—but with exponential stability replaced by ordinary stability (in Lyapunov's sense) in accordance with Definition 1, where $\delta = \delta(\varepsilon, \bar{\xi}) \to 0$ as $\varepsilon, \bar{\xi} \to 0$.

The same will be true when allowance is made for other non-ideal features of the control system. For example, if there is a small delay τ in the system measuring (estimating, observing) the state of the object, the controls M_i in (6.1) will have the form

$$M_i(t) = -H_i \operatorname{sign}[\dot{q}_i(t-\tau) - b_i(q(t-\tau), t)] \tag{6.3}$$

or, with a small time lag in the system drives

$$\gamma\dot{M}_i + M_i = H_i \operatorname{sign}[\dot{q}_i - b_i] \tag{6.4}$$

where $\gamma > 0$ is a sufficiently small number, etc. [14, 15].

When allowance is made for non-ideal features of the measuring and actuating elements of the control system, in the general case, the forces M_i may be written in the form

$$M_i = -H_i \operatorname{sign}(\chi_i) + Z_i(\chi_i), \quad \chi_i = \dot{q}_i - b_i \tag{6.5}$$

The quantities Z_i express the deviations of the non-ideal law (6.5) from the ideal one (3.1). Conditions stipulating that the effect of these non-ideal features of the measuring and actuating elements of the control system should be weak may be written for the general case in the form

$$N \leq \Delta, \quad \Delta = \text{const} > 0, \quad N = \sum_i Z_i\chi_i \tag{6.6}$$

The quantity N may be regarded as the total power of the forces Z_i perturbing the ideal control law (3.1) [14, 15]. Taking (6.5) into consideration, system (3.2) can be written in the form

$$\sum_r u_{sr}\ddot{\pi}_r = \sum_i (R_i - H_i \operatorname{sign}(\chi_i) + Z_i(\chi_i))b_{is} \tag{6.7}$$

Theorem 3. Under the assumptions of Theorem 1, suppose that the perturbing forces Z_i in (6.7) are small (the number Δ in (6.6) is small). Then any manifold (1.12) is a stable manifold of system (6.7).

The proof of Theorem 3 follows that of Theorem 1 and the well-known scheme of [10–15], except that instead of (4.17) we have the inequalities

$$\dot{G} \leq -\bar{\eta}\sqrt{G} + G(\bar{a} + \bar{b}\sqrt{G}) + \sum_s \dot{\pi}_s \sum_i Z_i b_{is} \tag{6.8}$$

$$\dot{G} \leq -\bar{\eta}\sqrt{G} + G(\bar{a} + \bar{b}\sqrt{G}) + \sum_i Z_i\chi_i \tag{6.9}$$

$$\dot{G} \leq -\bar{\eta}\sqrt{G} + G(\bar{a} + \bar{b}\sqrt{G}) + \Delta \tag{6.10}$$

The solutions of inequalities (6.10) have the form

$$G(t) \leq G(t^0), \quad t \geq t^0 \tag{6.11}$$

if the numbers $G(t^0)$ and Δ in (6.10) and (6.11) are sufficiently small. This implies the truth of Theorem 3.

7. ALLOWANCE FOR THE DRIVE DYNAMICS

Even when the dynamics of the drives of the system are significant (the numbers γ in (6.4) or Δ in (6.6) are not small), the manifold (1.12) of non-holonomic system (1.1)–(1.3) may still be stabilized. Let us consider the dynamics of the drives in the following form [11]

$$\dot{M}_i = F_i(M, q, \dot{q}, t) + U_i, \quad |U_i| \leq H_i \tag{7.1}$$

where U_i are new controls.

Differentiate Eqs (1.8)

$$\sum_r u_{sr}\dddot{\pi}_r = \sum_i (p_i(q, \dot{\pi}, \ddot{\pi}, t) + \dot{M}_i) b_{is} \tag{7.2}$$

and eliminate the quantities $\dot{M}_i$ using (7.1) and M_i using (1.8). The resulting system

$$\sum_r u_{sr}\dddot{\pi}_r = \sum_i (P_i + U_i) b_{is} \tag{7.3}$$

is equivalent to system (1.8), (7.1) [11].

We now introduce the set of unperturbed motions $(q, \dot{\pi}, \ddot{\pi}, t) = (q, 0, 0, t)$ of system (1.2), (1.3), (7.3) according to conditions (2.1)–(2.6) in the form

$$\sum_i (P_i^0 + U_i^0) b_{is} = 0 \tag{7.4}$$

$$\sum_i P_i^0 b_{is} = \sum_i U_i^0 b_{is}, \quad |U_i^0| \leq H_i \tag{7.5}$$

$$\left|\sum_i P_i^0 b_{is}\right| \leq \sum_i H_i |b_{is}| \tag{7.6}$$

$$\left|\sum_i P_i^0(q, 0, 0, t) b_{is}(q, t)\right| - \sum_i H_i |b_{is}(q, t)| \leq 0, \quad q \in \Phi \tag{7.7}$$

We define controls by analogy with (3.1)

$$U_i = -H_i \operatorname{sign}\left[\sum_s b_{is}(\ddot{\pi}_s + \lambda\dot{\pi}_s)\right] \tag{7.8}$$

where the number $\gamma = \text{const} > 0$ is sufficiently small. Then system (7.3) becomes

$$\sum_r u_{sr}\dddot{\pi}_r = \sum_i \left(P_i - H_i \operatorname{sign}\left[\sum_s b_{is}(\ddot{\pi}_s + \lambda\dot{\pi}_s)\right]\right) b_{is} \tag{7.9}$$

Theorem 4. Any manifold (1.12) is an exponentially stable manifold of system (1.2), (1.3), (7.9) (the definition is analogous to Definition 1) if manifold (1.12) is feasible in the sense that conditions (7.7) hold in the form

$$\left|\sum_i P_i^0(q, 0, 0, t)\right| \Bigg|_{q \in \Phi} \leq H_i - \eta \tag{7.10}$$

where the constant $\eta > 0$ may be fairly small and conditions (1.6) and (1.10) are satisfied.

The proof of Theorem 4 proceeds along the lines of Section 4 and will not be given here.

Theorems 1–4 indicate that the main assertion of this paper, as to the possibility of stabilizing manifolds for non-holonomic mechanical systems of general form, is of a non-local nature.

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